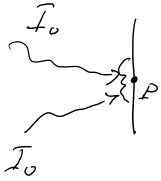
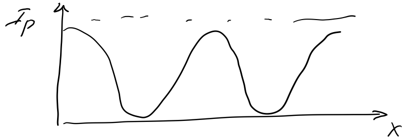
Multi-beam interference

So far we only considered the case of two interfering beams.



$I_1 = I_2 = I_0$

$I_p = 2I_0(1+\cos k\Delta)$

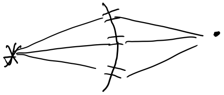


What happens if many beams will come to the point P?

$I_p = ?$

Multi-beam interference can be realized again by two methods:

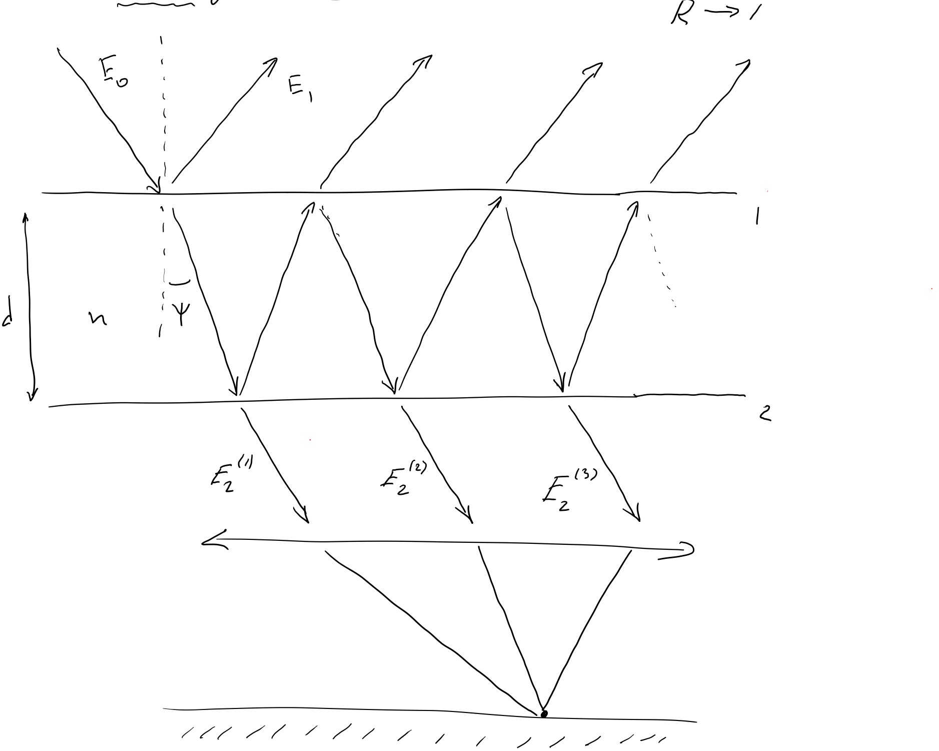
1) Method of wave front division (diffraction)



diffraction grating

2) Method of amplitude division

Fabry-Perot interferometer



Compared to interference model in thin film, here the surface reflection is $R \to 1$.

$E_2^{(1)}$, $E_2^{(2)}$ and $E_2^{(3)}$ are small as $R \to 1$.

$E_2^{(1)}$, $E_2^{(2)}$ and $E_2^{(3)}$ are coherent

① We need to show, that $E_2^{(1)}$, $E_2^{(2)}$ and $E_2^{(3)}$ are equal in intensity.

② We need to demonstrate, that optical path difference

$$\Delta\left(E_2^{(i+1)} - E_2^{(i)}\right) = 2dn\cos\psi$$

$$\delta = k_0\Delta = \frac{2\pi}{\lambda_0}\,2dn\cos\psi$$

phase difference

Let's introduce amplitude reflection coefficient

$$\rho = \frac{E_1}{E_0}\;\begin{array}{l} \rightsquigarrow \text{reflected amplitude} \\ \hookrightarrow \text{incident amplitude} \end{array}$$

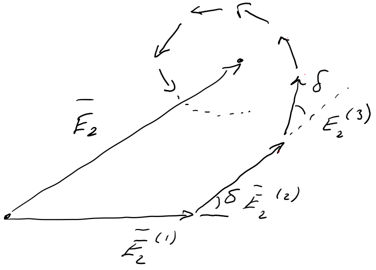
Amplitude transmission coefficient

$$\tau = \frac{E_2}{E_0}$$

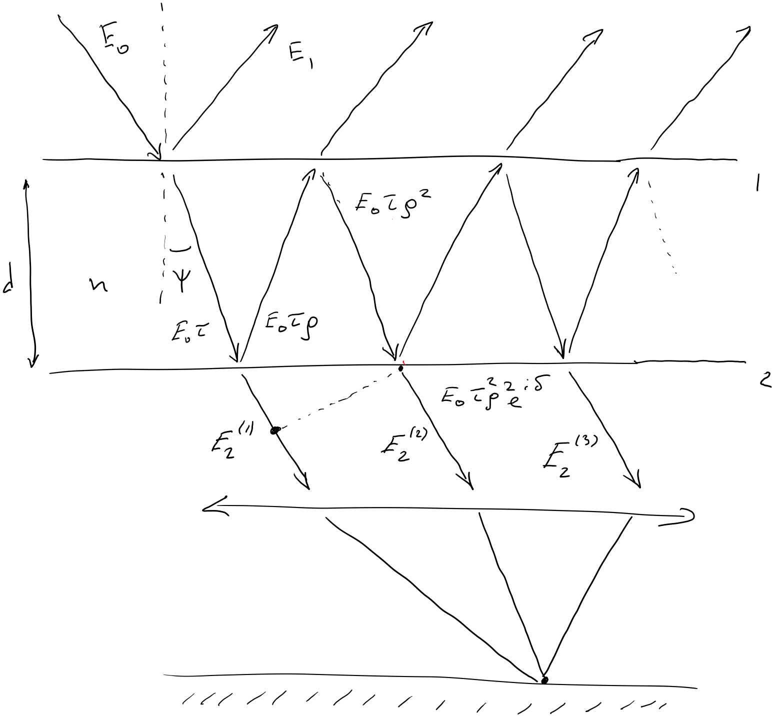
Transmitted field can be written:

$$E_2 = \sum_{i=1}^{\infty} E_2^{(i)}$$

We can use vector diagram:



Let's sum it up



$$\left.\begin{aligned}
E_2^{(1)} &= E_0\tau \\
E_2^{(2)} &= E_0\tau^2\rho^2 e^{i\delta} \\
E_2^{(3)} &= E_0\tau^2\rho^4 e^{i2\delta} \\
&\ldots\ldots \\
E_2^{(n)} &= E_0\tau^2\rho^{2(n-1)} e^{i(n-1)\delta}
\end{aligned}\right\} +$$

$$E_2 = E_0\tau^2\left(1+\rho^2 e^{i\delta}+\rho^4 e^{i2\delta}+\ldots+\rho^{2(n-1)} e^{i(n-1)\delta}+\ldots\right)$$

Σ is infinitly decreasing geometrical progression.

$$\Sigma = \frac{1}{1-q} \qquad q = \rho^2 e^{i\delta}$$

$$E_2 = E_0 \tau^2 \frac{1}{1-\rho^2 e^{i\delta}}$$

Let's move now to intensities

$$R = \rho^2 = \frac{I_1}{I_0}$$

$$T = \tau^2 = \frac{I_2}{I_0}$$

$$R + T = 1 \longrightarrow \text{no losses}$$

$$E_2 = E_0 \frac{T}{1 - Re^{i\delta}}$$

$$\frac{I_2}{I_0} = \frac{E_2 E_2^*}{I_0} = \frac{T^2}{(1-Re^{i\delta})(1-Re^{-i\delta})}$$

Let's simplify it!

$$(1-Re^{i\delta})(1-Re^{-i\delta}) = 1 + R^2 - 2R\cos\delta =$$

$$= 1 + R^2 - 2R\left(1 - 2\sin^2\frac{\delta}{2}\right) = (1-R)^2 +$$

$$+ 4R\sin^2\frac{\delta}{2}$$

$$\frac{I_2}{I_0} = \frac{(1-R)^2}{(1-R)^2 + 4R\sin^2\frac{\delta}{2}}$$

Let's introduce coefficient F:

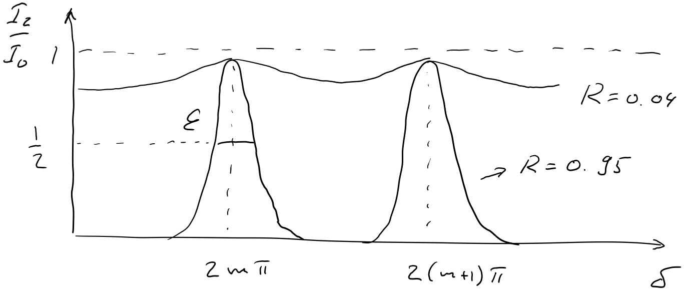
$$F = \frac{4R}{(1-R)^2}$$

$$\boxed{\frac{I_2}{I_0} = \frac{1}{1 + F \sin^2\frac{\delta}{2}}}$$ Airy equation

$$F = \begin{cases} \approx 0.2 & R = 0.04 \quad \text{air - glass} \\ \approx 1600 & R = 0.95 \quad \text{mirror} \end{cases}$$

$$\delta = k_0 \Delta = \frac{2\pi}{\lambda_0} 2 d n \cos\psi$$

If we are going to change incident angle φ, i.e. ψ, then we will have interference pattern.

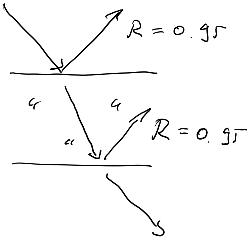


$$\left(\frac{I_2}{I_0}\right)_{max} \Rightarrow \delta = 2m\pi$$

It means that Airy function can reach 1

$$\frac{I_2}{I_0} = 1$$

How can we understand it if $R = 0.95$?



How ?

$$\left(\frac{I_2}{I_0}\right)_{min} = \frac{1}{1+F} = \begin{cases} \frac{1}{1.2} \approx 0.8 & \text{when } R = 0.04 \\ \frac{1}{1600} & \text{when } R = 0.95 \end{cases}$$

That is why we cannot observe interference on thin film in transmission geometry

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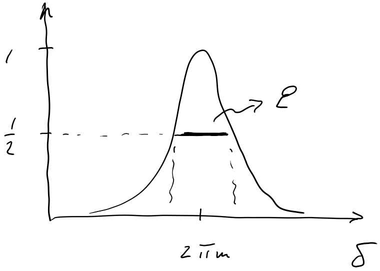
What is the width of our interference strips?

Let's find width ε



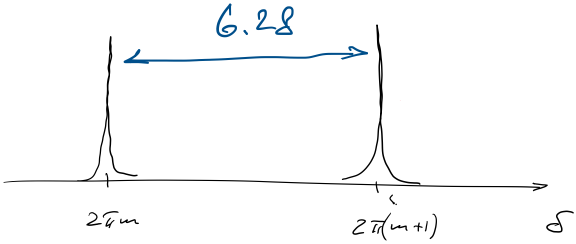
$$\frac{1}{2} = \frac{1}{1+F\sin^2\left(\frac{2\pi m+\frac{\varepsilon}{2}}{2}\right)} = \frac{1}{1+F\sin^2\frac{\varepsilon}{4}} \approx$$

$$\approx \frac{1}{1+F\left(\frac{\varepsilon}{4}\right)^2}$$

$$\varepsilon = \frac{4}{\sqrt{F}} = \frac{2(1-R)}{\sqrt{R}}$$

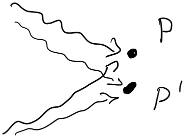
Hence, when $R = 0.95$ $\varepsilon = 0.1$.

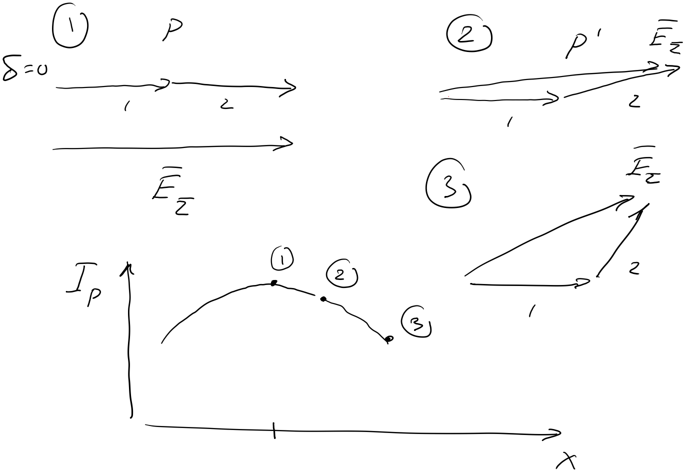
very narrow!



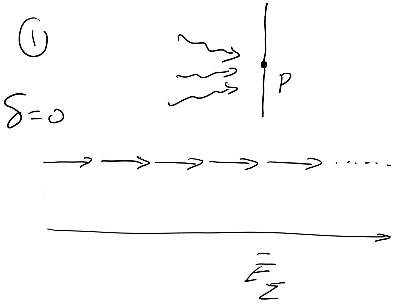
Why these interference lines are this narrow?

Let's first consider two beam interference.





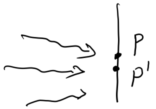
In case of multi-beam interference the problem is more sophisticated.



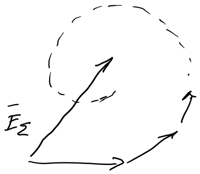
Estimate how many beams are interfering when $R = 0.95$

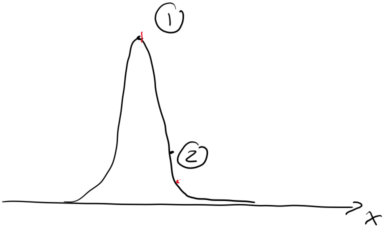
$E_2^{(n)} = 0.2 E_2^{(1)}$

②



Each vector will be shifted by δ in phase





Demonstration of Fabry-Perot interferometer

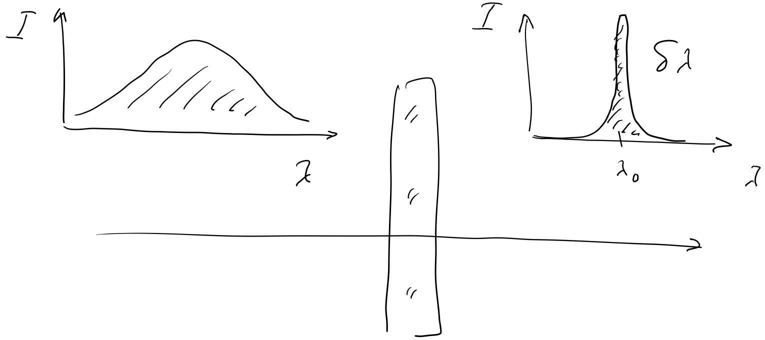
1) Na lamp

$\begin{cases} 589 \text{ nm} \\ 589.6 \text{ nm} \end{cases}$ Which line is closer to the center?

2) If we start to pump the air out, why do rings go closer to the center?

Interference filter

We can use this principle to transmit or reflect very narrow spectral band.



Demonstration

Let's explain this.

$$\delta = \frac{2\pi}{\lambda_0} 2dn\cos\varphi \qquad \varphi = 0$$

$$\delta = \frac{2\pi}{\lambda_0} 2dn$$

Condition for having maximum

$$\delta = 2\pi m$$

That will correspond to

$$\lambda_m = \frac{2dn}{m}$$

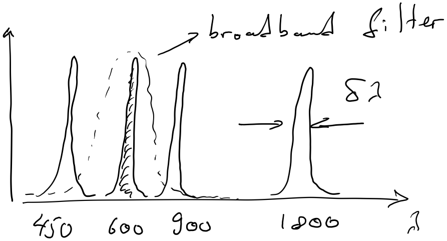
If we take $\lambda = 600$ nm and $m = 3$

$$2dn = 600 \cdot 3 = 1800 \text{ nm}$$

$$d = \frac{1800 \text{ nm}}{2n}$$

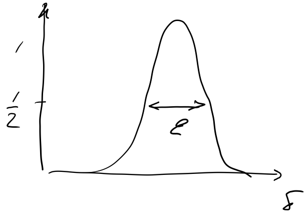
However, this filter will transmit other wavelength, that

$$\underbrace{2dn = m\lambda}_{} \Rightarrow \begin{array}{ll} 1800\,\text{nm} & (m=1) \\ 900\,\text{nm} & (m=2) \\ 450\,\text{nm} & (m=4) \end{array}$$



What spectral width our filter will have?

$\delta\lambda = ?$



$$\delta = \frac{2\pi}{\lambda_0} 2dn$$

$$\varepsilon \equiv d\delta = \frac{2\pi}{\lambda_0^2} 2dn\, \delta\lambda$$

$$\delta\lambda = \frac{\lambda_0^2 \varepsilon}{2\pi \underbrace{2dn}_{m\lambda_0}} = \frac{\lambda_0 \varepsilon}{2\pi m} = \frac{\varepsilon}{2\pi} \cdot \frac{\lambda_0}{m}$$

For our filter

$$\delta\lambda = \frac{\varepsilon}{2\pi} \cdot 200\,\text{nm}$$

$\varepsilon = 0.1$ $(R = 0.95)$

$$\delta\lambda = 3\,\text{nm}$$

$$\delta = \frac{2\pi}{\lambda_0} 2 d n \cos\psi = 2\pi m$$

Demonstration

Why maximum shifts towards shorter wavelength?